Power series everywhere convergent on $\mathbb{R}$ and all $\mathbb{Q}_p$

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Abstract. Power series are introduced that are simultaneously convergent for all real and p-adic numbers. Our expansions are in some aspects similar to those of exponential, trigonometric, and hyperbolic functions. Starting from these series and using their factorial structure new and summable series with rational sums are obtained. For arguments $x \in \mathbb{Q}$ adeles of series are constructed. Possible applications at the Planck scale are also considered.

1. INTRODUCTION

The field of rational numbers is of central importance in physics and mathematics. It is well known that all results of measurements belong to $\mathbb{Q}$, i.e. that the irrational numbers cannot be measured. From a mathematical point of view $\mathbb{Q}$ is the simplest infinite number field. Completion of $\mathbb{Q}$ with respect to the absolute value gives the field of real numbers $\mathbb{R}$. Algebraic closure of $\mathbb{R}$ leads to the field of complex numbers $\mathbb{C}$. Although experimental results are given in $\mathbb{Q}$, theoretical models are usually constructed over $\mathbb{R}$ or $\mathbb{C}$. Comparison between theory and experimental results performs within $\mathbb{Q}$.

However, it is interesting that in addition to the standard absolute value there exist p-adic norms (valuations) on $\mathbb{Q}$. Completions of $\mathbb{Q}$ with respect to p-adic norms give us the fields of p-adic numbers $\mathbb{Q}_p$ ($p =$ a prime number). There is also a p-adic analog of the complex numbers. According to this similarity between p-adic and real numbers, it is natural to expect that p-adic numbers should also play a significant role in theoretical and mathematical physics.

Since 1987, p-adic numbers have been successfully considered in string theory [1], quantum mechanics [2], quantum field theory [3], and in some other branches of theoretical [4, 5] and mathematical [6] physics. Such new theoretical constructions are p-adic analogs of some models on real (or complex) numbers.

There has been also a research on various p-adic aspects of the perturbation series [7]. It is shown that the usual perturbation series, which are divergent in the real case, are p-adic convergent. Summability of a given series in all but a finite number of $\mathbb{Q}_p$ may be used for summation of a divergent counterpart at the rational points.

In order to make a direct connection of p-adic models with the real one it seems to be necessary to have convergence in $\mathbb{R}$ and all $\mathbb{Q}_p$ within the common domain of rational numbers. However, the standard power series of theoretical physics do not satisfy this

property. For example, expansions of functions $\exp x, \sin x, \cos x, \sinh x$, and $\cosh x$ are convergent in the p-adic case for $|x|_p < 1$ if $p \neq 2$ and $|x|_2 < \frac{1}{2}$. As a consequence, there is no $0 \neq x \in \mathbb{Q}_p$ for which these functions are defined for any p.

This paper is devoted to the power series that converge everywhere on $\mathbb{R}$ and everywhere on $\mathbb{Q}_p$ for every p. These analytic functions are simple and suitable modifications of expansions for exponential, trigonometric, and hyperbolic functions.

An appropriate mathematical background on p-adic numbers and p-adic analysis can be found in Refs. [8]-[10].

2. EVERYWHERE CONVERGENT SERIES

In theoretical physics we often encounter a power series

$$\sum_{n=0}^{\infty} A_n x^n \,, \tag{1}$$

where $A_n \in \mathbb{Q}$ and $x \in \mathbb{Q}$. If we take $x \in \mathbb{Q}_p$ series (1) may be treated as the p-adic one. It is obvious that the domain of convergence for any of the number fields depends on coefficients A_n. One usually has that to large (small) radius of convergence in the real case corresponds the small (large) one in the p-adic case. In order to improve this situation we shale make an appropriate modification of some elementary functions. By virtue of the simplicity and enormous applications in overall theoretical and mathematical physics we shall concentrate our attention on the exponential, trigonometric, and hyperbolic functions.

Recall that the series

$$\varphi^{\epsilon}_{\mu,\nu}(x) = \sum_{n=0}^{\infty} \epsilon^n \, \frac{x^{\mu n+\nu}}{(\mu n+\nu)!} \tag{2}$$

contains the following functions: $\exp x \, (\epsilon = 1,\, \mu = 1,\, \nu = 0)$, $\cos x \, (\epsilon = -1,\, \mu = 2,\, \nu = 0)$, $\sin x \, (\epsilon = 1,\, \mu = 2,\, \nu = 1)$, $\cosh x \, (\epsilon = 1,\, \mu = 2,\, \nu = 0)$, and $\sinh x \, (\epsilon = 1,\, \mu = 2,\, \nu = 1)$. It is well known from classical analysis that series (2) is everywhere convergent on $\mathbb{R}$.

Theorem 1: Power series

$$\Phi^{\epsilon,q}_{\mu,\nu}\,(x) = \sum_{n=0}^{\infty} \epsilon^n \, I^{(q)}_{\mu n+\nu} \frac{x^{\mu n+\nu}}{(\mu n+\nu)!} \,, \tag{3}$$

where $\epsilon \pm 1$, $0 < q \in \mathbb{Q}$, $\mu \in \mathbb{N}$, $\nu \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and

$$I^{(q)}_{\mu n+\nu} = \frac{((\mu n+\nu)!)^{\mu n+\nu}}{q + ((\mu n+\nu)!)^{\mu n+\nu}} \tag{4}$$

converges for all $x \in \mathbb{R}$ and all $x \in \mathbb{Q}_p$ for every p.

Proof: In a real case the above theorem follows from the fact that for large enough n parameter q can be neglected in comparison to the factorial term and $I^{(q)}_{\mu n+\nu}$ may be approximated by 1. Hence series (3) asymptotically behaves like (2) which is convergent

at all real x. Recall [8] that, in p-adic case, a necessary and sufficient condition for a convergence of (1) is

$$\mid A_n x^n \mid_p \to 0, \quad \text{as} \quad n \to \infty. \tag{5}$$

As a consequence of (5) it is enough to consider the p-adic norm of the general term in (3), i.e.,

$$\Big| \epsilon^n I^{(q)}_{\mu n+\nu} \frac{x^{\mu n+\nu}}{(\mu n+\nu)!} \Big|_p = \frac{|(\mu n+\nu)!|_p^{\mu n+\nu-1}}{|q+((\mu n+\nu)!)^{\mu n+\nu}|_p} \, |x|_p^{\mu n+\nu}. \tag{6}$$

Since the p-adic norm satisfies the strong triangle inequality, one has

$$|q+((\mu n+\nu)!)^{\mu n+\nu}|_p = |q|_p \tag{7}$$

for large enough n. Note that

$$|n!|_p = p^{-(n-n')/(p-1)}, \tag{8}$$

where n' is the sum of digits in the canonical expansion of n over p. According to (8), one has, for the numerator of (6),

$$|(\mu n+\nu)!|_p^{\mu n+\nu-1} \, |x|_p^{\mu n+\nu} = (p^{-\{[\mu n+\nu-(\mu n+\nu)']/(p-1)\}[(\mu n+\nu-1)/(\mu n+\nu)]} \, |x|_p)^{\mu n+\nu}_{n\to\infty} \to 0\,, \tag{9}$$

which is valid for any p and all $x \in \mathbb{Q}_p$. On the basis of (7) and (9) it follows everywhere convergence on $\mathbb{Q}_p$ for any p. Thus Theorem 1 is proved.

Among all possible examples of analytic functions contained in power series (3) we want to point out the following ones:

$$\exp_q x = \sum_{n=0}^{\infty} \frac{(n!)^n}{q+(n!)^n} \, \frac{x^n}{n!} \tag{10a}$$

$$\cos_q x = \sum_{n=0}^{\infty} (-1)^n \, \frac{((2n)!)^{2n}}{q+((2n)!)^{2n}} \, \frac{x^{2n}}{(2n)!} \tag{10b}$$

$$\sin_q x = \sum_{n=0}^{\infty} (-1)^n \, \frac{((2n+1)!)^{2n+1}}{q+((2n+1)!)^{2n+1}} \, \frac{x^{2n+1}}{(2n+1)!} \tag{10c}$$

$$\cosh_q x = \sum_{n=0}^{\infty} \frac{((2n)!)^{2n}}{q+((2n)!)^{2n}} \, \frac{x^{2n}}{(2n)!} \tag{10d}$$

$$\sinh_q x = \sum_{n=0}^{\infty} \frac{((2n+1)!)^{2n+1}}{q+((2n+1)!)^{2n+1}} \, \frac{x^{2n+1}}{(2n+1)!} \tag{10e}$$

Inverse functions of (10a)-(10e) can be defined in the usual way, where coefficients in the power expansions are appropriately modified. For example,

$$y = \ln_q x = \sum_{n=1}^{\infty} (-1)^{n+1} \, a_n^{(q)} \, \frac{(x - I_0^{(q)})^n}{n},$$

where $I_0^{(q)} = (q+1)^{-1}$ and $a_1^{(q)} = (I_0^{(q)})^{-1} = q+1, \ a_2^{(q)} = I_2^{(q)}/(I_1^{(q)})^3 = 4(q+1)^3/(q+4), \cdots$.

3. ADELIC ASPECTS

Recall [10] that an adele is an infinite sequence

$$a = (a_\infty,\, a_2,\, \cdots, a_p,\, \cdots), \tag{10}$$

where $a_\infty \in \mathbb{Q}_\infty = \mathbb{R}$, $a_p \in \mathbb{Q}_p$ with the restriction that all but a finite number of $a_p \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p |\, |x|_p \leq 1\}$. The set of adeles $\mathbb{A}$ is a ring under componentwise addition and componentwise multiplication. It is an additive group $\mathbb{A}^+$ with respect to addition. The subset of $\mathbb{A}$ with $\lambda_\infty \neq 0$, $\lambda_p \neq 0$ for all p, and $|\lambda_p|_p = 1$ for all but a finite number of p is a multiplicative group of ideles $\mathbb{A}^*$. One has a principal adele (idele) if

$$r = (r,\, r,\, \cdots, r,\, \cdots), \tag{11}$$

where $r \in \mathbb{Q}$ $(r \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\})$. One can define a product of norms on ideles

$$|\lambda| = |\lambda_\infty|_\infty \prod_p |\lambda_p|_p\,, \tag{12}$$

where $|\cdot|_\infty$ denotes the usual absolute value. For a principal idele it yields

$$|r| = |r|_\infty \prod_p |r|_p = 1. \tag{13}$$

Equation (13) is a well-known product formula for nonzero rational numbers. An additive character on $\mathbb{A}^+$ is

$$\begin{aligned} \chi_b(a) &= \exp 2\pi i(-a_\infty b_\infty + a_2 b_2 + \cdots + a_p b_p + \cdots) \\ &= \exp(-2\pi i\, a_\infty b_\infty) \prod_p \exp 2\pi i\, \{a_p b_p\}_p\,, \end{aligned} \tag{14}$$

where $a, b \in \mathbb{A}^+$, and $\{x_p\}_p$ denotes a fractional part of x_p. On an idele,

$$\lambda = (\lambda_\infty, \lambda_2, \cdots, \lambda_p, \cdots), \tag{15}$$

there exists multiplicative character

$$\pi(\lambda) = \pi_\infty(\lambda_\infty)\, \pi_2(\lambda_2) \cdots \pi_p(\lambda_p) \cdots \;= |\lambda_\infty|_\infty^{c_\infty} \prod_p |\lambda_p|_p^{c_p}, \tag{16}$$

where c_∞ and c_p are complex numbers. Note that in (14) and (16) only finitely many factors are different from unity.

It may be of physical interest to construct adeles from series (3) while their arguments x belong to the principal adeles (11).

Theorem 2: Let us have a sequence,

$$\Phi^\epsilon_{\mu,\nu}(x) = (\varphi^\epsilon_{\mu,\nu}(x),\, \Phi^{\epsilon,1/2}_{\mu,\nu}(x),\, \cdots, \Phi^{\epsilon,1/p}_{\mu,\nu}(x), \cdots), \tag{17}$$

where $\varphi^\epsilon_{\mu,\nu}(x)$ is a real series defined by (2) , and $\Phi^{\epsilon,1/p}_{\mu,\nu}(x)$ is a p-adic series defined by (3) If $x = r \in \mathbb{Q}$ then (17) is an adele.

Proof: There is no problem with real function $\varphi^\epsilon_{\mu,\nu}(x)$ for any $x \in \mathbb{Q}$. The general term of the p-adic series (3) for $q = 1/p$ is

$$\epsilon^n\, \frac{((\mu n+\nu)!)^{\mu n+\nu-1}}{1/p + ((\mu n+\nu)!)^{\mu n+\nu}}\, x^{\mu n+\nu} = \epsilon^n\, \frac{p\,((\mu n+\nu)!)^{\mu n+\nu-1}}{1 + p\,((\mu n+\nu)!)^{\mu n+\nu}}\, x^{\mu n+\nu}\,. \tag{18}$$

It is obvious that $|1 + p\,((\mu n+\nu)!)^{\mu n+\nu}|_p = 1$. Hence the p-adic norm of (18) is

$$\frac{1}{p}|((\mu n+\nu)!)^{\mu n+\nu-1}|_p\,|x|_p^{\mu n+\nu}\,, \tag{19}$$

which, for a given $x = r$, can be larger than 1 only for a finite number of p. Since this conclusion is valid for any $\mu \in \mathbb{N}$ and ν, $n \in \mathbb{N}_0$ it follows that $|\Phi_{\mu,\nu}^{\epsilon,1/p}(r)|_p \geq 1$ only for a finite number of p. So, it is shown that $\Phi_{\mu,\nu}^{\epsilon,1/p}(x)$ is an adele when x is a principal adele.

Note that instead of $\varphi_{\mu,\nu}^{\epsilon}(r)$, which is $\Phi_{\mu,\nu}^{\epsilon,0}(r)$, one can take for the real term in (17) any of the series $\Phi_{\mu,\nu}^{\epsilon,q}(r)$ defined by (3). It is easy to see that $1/p$ in (17) can be substituted by p^{-s}, where $s \in \mathbb{N}$.

4. ON SUMMATION

It is not clear that there does exist $0 \neq x\mathbb{Q}$ for which series (3) is a rational number. By an analogy with the real case of series (2) one can expect that there is not such a possibility. For a trivial case, i.e. $x = 0$, one has

$$\Phi_{\mu,\nu}^{\epsilon,q}(0) = \begin{cases} 0, & \nu \geq 1, \\ \frac{1}{q+1}, & \nu = 0. \end{cases} \tag{20}$$

We shall show that starting from series (3) one can obtain a sum of the corresponding functional series.

Theorem 3: The summation formula,

$$\sum_{n=0}^{\infty} ((\mu n+\nu)!)^{\mu n+\nu-1}\, x^{\mu n}\Big\{\frac{[(\mu(n+1)+\nu)!]^{\mu}\,(\mu n+\nu+1)_{\mu}^{\mu n+\nu-1}}{q+[(\mu(n+1)+\nu)!]^{\mu(n+1)+\nu}}\,x^{\mu} - \frac{1}{q+((\mu n+\nu)!)^{\mu n+\nu}}\Big\} = -\frac{(\nu!)^{\nu-1}}{q+(\nu!)^{\nu}}\,, \tag{21}$$

where $(\mu n+\nu+1)_{\mu} = (\mu n+\nu+1)\,(\mu n+\nu+2)\cdots(\mu n+\nu+\mu)$, has a place for all $0 \neq x \in \mathbb{R}$ as well as for all $0 \neq x \in \mathbb{Q}_p$ for every p.

Proof: Expansion (3) of $\Phi_{\mu,\nu}^{-1,q}(x)$ can be rewritten as follows:

$$\Phi_{\mu,\nu}^{-1,q}(x) = \frac{(\nu!)^{\nu-1}}{q+(\nu!)^{\nu}}\,x^{\nu} + ((\mu+\nu)!)^{\mu+\nu-1}\,x^{\mu+\nu}\Big[\frac{((2\mu+\nu)!)^{\mu}\,(\mu+\nu+1)_{\mu}^{\mu+\nu-1}}{q+((2\mu+\nu)!)^{2\mu+\nu}}\,x^{\mu} - \frac{1}{q+((\mu+\nu)!)^{\mu+\nu}}\Big] + \cdots\,, \tag{22}$$

$$-\,\Phi_{\mu,\nu}^{-1,q}(x) = (\nu!)^{\nu-1}\,x^{\nu}\,\Big[\frac{((\mu+\nu)!)^{\mu}\,(\nu+1)_{\mu}^{\nu-1}}{q+((\mu+\nu)!)^{\mu+\nu}}\,x^{\mu} - \frac{1}{q+(\nu!)^{\nu}}\Big] + ((2\mu+\nu)!)^{2\mu+\nu-1} \times x^{2\mu+\nu}\,\Big[\frac{((3\mu+\nu)!)^{\mu}\,(2\mu+\nu+1)_{\mu}^{2\mu+\nu-1}}{q+((3\mu+\nu)!)^{3\mu+\nu}}\,x^{\mu} - \frac{1}{q+((2\mu+\nu)!)^{2\mu+\nu}}\Big] + \cdots. \tag{23}$$

Addition of (22) and (23) with division by x^{ν} for $x \neq 0$ leads to formula (21).

In the particular case ($\mu = 1$, $\nu = 0$, $x = -1$, $q = 1$), we have

$$\sum_{n=0}^{\infty}(-1)^n\,\frac{((n+1)!)^n\,[1+(2+n)(n!)^n] + (n!)^{n-1}}{[1+(n!)^n]\,[1+((n+1)!)^{n+1}]} = \frac{1}{2}, \tag{24}$$

which is the result valid in $\mathbb{R}$ and all $\mathbb{Q}_P$.

5. DISCUSSION AND CONCLUDING REMARKS

The first question that we want to discuss is related to the possible applications of the power series (3) . Recall that the interest in p-adic models is mainly motivated by some indications [1] that space-time at the Planck scale should be analyzed using p-adic mathematics. According to this point of view, let us consider classical cosmological solutions of the Einstein gravitational equations. These equations for the scale factor $R(t)$ of the homogeneous and isotropic universe are

$$\frac{\ddot{R}(t)}{R(t)} = -\frac{\kappa\,(\rho + 3p)}{6}\,, \quad \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{\kappa\,\rho}{3}\,, \tag{25}$$

where ρ is the energy density and p is the corresponding pressure ($\kappa = 8\pi G$; $k = +1,\,-1,\,0$). Assume that ρ and p depend on time in the way that enables an application of (3) for $R(t)$. Among the three cases let us choose the following one:

$$\begin{aligned} &k = 0\,, \\ &\rho_q(t) = \frac{3}{\kappa}\left[\frac{d}{dt}\ln(\exp_q Ht)\right]^2\,, \\ &p_q(t) = -\rho_q(t) - \frac{2}{\kappa}\frac{d^2}{dt^2}\ln(\exp_q Ht)\,, \\ &R_q(t) = H^{-1}\exp_q Ht, \quad H = \sqrt{\Lambda/3}. \end{aligned} \tag{26}$$

An analogous situation is for $k = 1$ with $R(t) = H^{-1}\cosh_q Ht$ and $k = -1$ with $R(t) = H^{-1}\sinh_q Ht$. If $\kappa \in \mathbb{Q}$ all theses models may be treated either real or p-adic. By decreasing parameter q, expansion of the universe given by (26) can be done arbitrary close to the de Sitter model. Namely, when $q \to 0:\ \rho_q \to \Lambda/\kappa,\ p_q \to -\Lambda/\kappa$, and $R_q(t) \to R(t) = \sqrt{3/\Lambda}\exp\sqrt{\Lambda/3}\,t$, where Λ is the cosmological constant. It would be interesting to find a scalar-field model that leads to $\rho_q(t)$ and $p_q(t)$. A classical cosmological solution for $k = +1$ and $R_q(t) = \sqrt{3/\Lambda}\cosh_q\sqrt{\Lambda/3}\,t$ can be further used in p-adic quantum cosmology [5], which is a generalization of the Hartle-Hawking approach to the wave function of the universe.

Let us suppose that the parameter q is a quotient ($q = l_{Pl}/l$) of the Planck length ($l_{Pl} \sim 10^{-33}$ cm) and a length that characterizes the given scale. For example, the unification length in the electroweak theory is about 10^{-17}cm and for the GUT one gets $l \sim 10^{-29}$cm. In such a way only at the Planck scale parameter q ($q = 1$) cannot be neglected. So, $q = 1$ in the real case and $q = 1/p$ in the p-adic case should be natural values in the high-energy limit ($E \sim 10^{19}$ GeV) . Taking $\Phi^{\epsilon,1}_{\mu,\nu}(x)$ instead of $\varphi^{\epsilon}_{\mu,\nu}(x)$ in (17), one has a unification at the Planck scale of real and p-adic functions (3) in the form of adeles.

Parameter q regularizes (2) to enlarge the region of convergence from $|x|_p < 1$, $p \neq 2$ (and $|x|_2 < \frac{1}{2}$) to the whole $\mathbb{Q}_p$. In the limit $q \to 0$ the regularized functions tend to the usual ones.

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